Junior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

26 – 29 April 2021

These solutions augment the shorter solutions also available online. The solutions given here are full solutions, as explained below. In some cases we give alternative solutions. There are also many additional problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that occasionally you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can sometimes be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, sometimes, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	E	A	D	E	B	C	E	E	C	C	D	B	A	D	B	C	C	D	E	B	A	E	D	B

1. What is the value of $123 - 456 + 789$?

A 456 B 556 C 567 D 678 E 789

Solution **A**

Commentary

Because $123 - 456$ is a negative number, this calculation is a little awkward without a calculator. We can get round this difficulty by rearranging the order of the numbers, so that we don't have to subtract a larger number from a smaller number.

We have

$$\begin{aligned} 123 - 456 + 789 &= 123 + 789 - 456 \\ &= 912 - 456 \\ &= 456. \end{aligned}$$

For investigation

1.1 Find the values of

(a) $222 - 333 + 444$,

(b) $3333 - 4444 + 5555$.

1.2 Solve the equation $321 - x + 123 = 222$.

2. Brianna has £20, all in 5p coins, and £50, all in 2p coins.

How many coins does she have in total?

A 200 B 290 C 1000 D 2540 E 2900

Solution **E**

£20 is 2000p. Since $2000 \div 5 = 400$, the number of 5p coins that Brianna has is 400.

£50 is 5000p. Since $5000 \div 2 = 2500$, the number of 2p coins that Brianna has is 2500.

Therefore, in total, Brianna has $400 + 2500 = 2900$ coins.

For investigation

2.1 What is the mean value of the 2p and 5p coins that Brianna has?

2.2 How many 10p coins would Brianna need to have in addition to her 2p and 5p coins so that the mean value of her 2p, 5p and 10p coins was 5p?

2.3 Brianna has just 2p and 5p coins. She has a total of 400 coins which together are worth £17.60. How many 5p coins does she have?

3. What is the value of $1 - 2 \times 3 + 4 \div 5$?

A -4.2 B -2.8 C 0 D 0.2 E 4

SOLUTION **A**

COMMENTARY

It is important to remember that the multiplication and division must be done before the subtraction and the addition.

This convention is used to avoid the need to write expressions, such as the one in this question, with brackets to indicate the order in which the operations should be carried out. For example, here we don't need to write the expression as $1 - (2 \times 3) + (4 \div 5)$.

We have

$$\begin{aligned} 1 - 2 \times 3 + 4 \div 5 &= 1 - 6 + 0.8 \\ &= -5 + 0.8 \\ &= -4.2. \end{aligned}$$

FOR INVESTIGATION

3.1 Find the value of the following.

(a) $1 \times 2 + 3 \times 4 + 5 \times 6$.

(b) $1 \div 2 + 3 \div 4 + 5 \div 6$.

(c) $1 - 2 \div 3 + 4 \times 5$.

3.2 (a) In the equation

$$1 + 2 \diamond 3 + 4 \diamond 5 = 27$$

the symbol $\diamond$ represents one of the mathematical operations $+$, $-$, $\times$, $\div$ (the same operation in both places).

Which choice of these operations for $\diamond$ makes the equation correct?

(b) In the equation

$$1 \heartsuit (2 \heartsuit 3) + 4 \heartsuit 5 = 1$$

the symbol $\heartsuit$ represents one of the mathematical operations $+$, $-$, $\times$, $\div$ (the same operation in all three places).

Which choice of these operations for $\heartsuit$ makes the equation correct?

4. How many of the following numbers are multiples of 11?

187	156	253	495	132

A 1 B 2 C 3 D 4 E 5

Solution **D**

Method 1

We see that

$$\begin{aligned} 187 &= 11 \times 17, \\ 156 &= 11 \times 14 + 2, \\ 253 &= 11 \times 23, \\ 495 &= 11 \times 45, \\ \text{and} \quad 132 &= 11 \times 12. \end{aligned}$$

Therefore of the given numbers only 156 is not a multiple of 11. Hence four of them are multiples of 11.

Method 2

In this method we use the following fact:

The remainder when a three-digit integer 'cba' is divided by 11 is the same as the remainder when the integer $(a + c) - b$ is divided by 11. [This is a particular case of a more general test for divisibility by 11 that is discussed below in Problem 4.2.]

Applying this test we find that we have

$$\begin{aligned} 187 &: \quad (7 + 1) - 8 = 0, \\ 156 &: \quad (6 + 1) - 1 = 2, \\ 253 &: \quad (2 + 3) - 5 = 0, \\ 495 &: \quad (5 + 4) - 9 = 0, \\ \text{and} \quad 132 &: \quad (2 + 1) - 3 = 0. \end{aligned}$$

We deduce that in four cases the remainder when the given number is divided by 11 is 0. Hence four of the given numbers are multiples of 11.

For investigation

4.1 Show that the remainder when the three-digit number 'cba' is divided by 11 is the same as the remainder when $(a + c) - b$ is divided by 11.

4.2 Show that the remainder when an n-digit integer '$\ldots fedcba$' is divided by 11 is the same as the remainder when $(a + c + e + \ldots) - (b + d + f + \ldots)$ is divided by 11.

4.3 What is the remainder when 98 765 432 123 456 789 is divided by 11?

5. When I have walked 20% of the way to school, I have 1200 metres more to walk than when I have 20% of the walk remaining.

How far, in metres, is it from my home to my school?

A 1240 B 1440 C 1680 D 1800 E 2000

SOLUTION **E**

The first and last parts of the walk are each 20% of the total distance.

So the middle portion , which is 1200 metres, is 60% of the walk.

Since 60% of the journey is 1200 metres, 1% of the walk is $(1200 \div 60)$ metres, that is 20 metres.

Hence 100% of the walk is 100×20 metres, that is, 2000 metres.

6. What is the value of $(2 - \frac{1}{2})(3 - \frac{1}{3})(4 - \frac{1}{4})$?

A 16 B 15 C 14 D 13 E 12

SOLUTION **B**

We have

$$2 - \tfrac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2},$$
$$3 - \tfrac{1}{3} = \frac{9}{3} - \frac{1}{3} = \frac{8}{3},$$

and

$$4 - \tfrac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

Therefore

$$\begin{aligned}(2 - \tfrac{1}{2})(3 - \tfrac{1}{3})(4 - \tfrac{1}{4}) &= \frac{3}{2} \times \frac{8}{3} \times \frac{15}{4} \\ &= 15.\end{aligned}$$

FOR INVESTIGATION

6.1 What is the value of

$$(2 - \tfrac{1}{2})(3 - \tfrac{1}{3})(4 - \tfrac{1}{4})(5 - \tfrac{1}{5})\ ?$$

6.2 Which is the least positive integer n such that

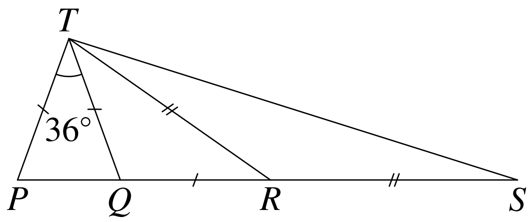
$$(2 - \tfrac{1}{2})(3 - \tfrac{1}{3})(4 - \tfrac{1}{4}) \dots (n - \tfrac{1}{n}) > 1\,000\,000\,?$$

7. In the diagram shown, $PT = QT = QR$.

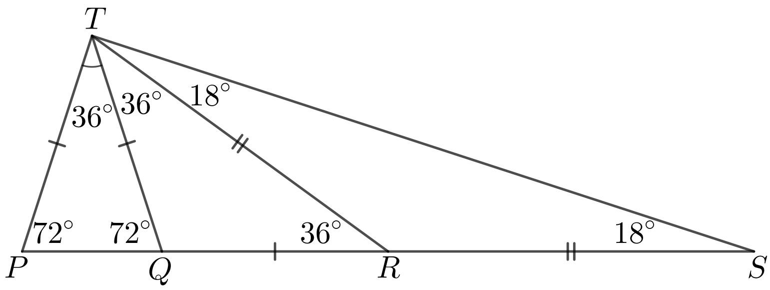
Also, $RT = RS$ and $\angle PTQ = 36°$.

What is $\angle PTS$?

A 72° B 80° C 90°
D 100° E 108°



Solution **C**



Because $PT = QT$, the triangle PQT is isosceles and hence $\angle TPQ = \angle TQP$.

Because the angles in a triangle have sum 180°, $\angle TPQ + \angle TQP + \angle PTQ = 180°$.

Therefore $2\angle TQP + 36° = 180°$. Hence $\angle TQP = \frac{1}{2}(180 - 36)° = \frac{1}{2}(144)° = 72°$.

Because $QR = QT$, we have $\angle QRT = \angle QTR$.

By the External Angle Theorem [see Problem 7.1] applied to the triangle TQR, $\angle QRT + \angle QTR = \angle TQP = 72°$.

It follows that $\angle QTR = \angle QRT = 36°$.

Similarly in the triangle TRS, we have $RS = RT$, and hence $\angle RST = \angle RTS$, and by the External Angle Theorem, $\angle RST + \angle RTS = \angle QRT = 36°$.

Therefore $\angle RST = \angle RTS = 18°$.

We can now deduce that

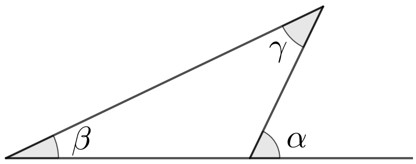
$$\angle PTS = \angle PTQ + \angle QTR + \angle RTS = 36° + 36° + 18° = 90°.$$

For investigation

7.1 The *External Angle Theorem* says that the external angle of a triangle is the sum of the two opposite internal angles.

In terms of the diagram it says that $\alpha = \beta + \gamma$.

Explain why the External Angle Theorem is true.



7.2 Suppose that in the diagram of the question $\angle PTQ = 20°$. What is $\angle PTS$?

7.3 Suppose that in the diagram of the question $\angle PTQ = x°$. Find a formula for $\angle PTS$ in terms of x.

8. What is the value of $1 - (2 - (3 - (4 - 5)))$?

A -5 B -3 C -1 D 1 E 3

Solution **E**

We have

$$\begin{aligned} 1 - (2 - (3 - (4 - 5))) &= 1 - (2 - (3 - (-1))) \\ &= 1 - (2 - (3 + 1)) \\ &= 1 - (2 - 4) \\ &= 1 - (-2) \\ &= 1 + 2 \\ &= 3. \end{aligned}$$

For investigation

8.1 What is the value of $1 - (2 - (3 - (4 - (5 - 6))))$?

8.2 What is the value of $1 - (2 - (3 - (4 - (5 - (6 - 7)))))$?

8.3 Find a formula in terms of n for the value of

$$1 - (2 - (3 - (4 - (5 - (6 - (7 - (\cdots - n)))))) \ldots)$$

in the case where n is an odd positive integer.

8.4 Find a formula in terms of n for the value of

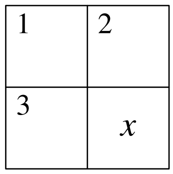
$$1 - (2 - (3 - (4 - (5 - (6 - (7 - (\cdots - n)))))) \ldots)$$

in the case where n is an even positive integer.

9. Each cell in the crossnumber below contains a single non-zero digit. The answer to each clue is a two-digit number.

Clues

Across	Down
1. A square	1. A square
3. An odd square	2. A square



What is the value of x?

A 1 B 3 C 5 D 7 E 9

Solution **E**

The answer to 3 Across is a two-digit odd square. Hence it is either 25, 49 or 81.

The tens digit of 3 Across is the units digit [sometimes called the *ones digit*] of the answer to 1 Down which is a square.

The units digit of a square cannot be 2 or 8. Hence 3 Across is neither 25 nor 81. Therefore it is 49. Hence x is 9.

For investigation

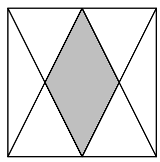
9.1 Explain why the units digit of a square cannot be 2 or 8.

9.2 Which other digits cannot be the units digit of a square?

9.3 Complete the crossnumber.

10. The diagram shows a rhombus formed by joining each vertex of a square to the midpoint of a side of the square.

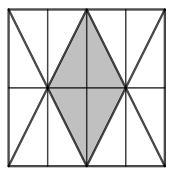
What fraction of the area of the square has been shaded?



A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{6}$ E $\frac{1}{8}$

Solution **C**

The diagram on the right shows the square divided into sixteen congruent triangles.



Four of these sixteen triangles are shaded.

Therefore the fraction of the square that has been shaded is $\frac{4}{16}$, that is, $\frac{1}{4}$.

For investigation

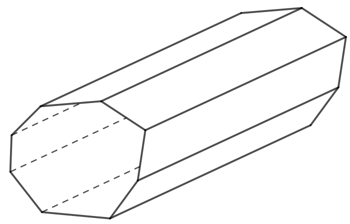
10.1 Explain why the sixteen triangles into which the square has been divided are congruent.

11. A particular prism has ten faces. How many edges does it have?

A 20 B 21 C 24 D 27 E 30

Solution **C**

Suppose that a prism has end faces that are congruent polygons with n edges. Note that these do not need to be regular polygons.



In addition to these two end faces the prism has n rectangular faces joining the end faces as shown in the diagram. Therefore the prism has a total of $n + 2$ faces. The prism in this question has 10 faces. Therefore in this case $n = 8$.

So the two end polygons each have 8 edges. There are also 8 edges joining the two faces. Therefore the total number of edges is $2 \times 8 + 8 = 24$.

For investigation

11.1 A particular pyramid has twelve edges. How many faces does it have?

11.2 The great Swiss mathematician Leonhard Euler (1707-1783) found the famous formula

$$V + F = E + 2$$

for convex polyhedra. In this formula, V is the number of vertices, E is the number of edges, and F is the number of faces.

Check that, for every positive integer n greater than 2, Euler's formula is true for a prism whose end faces are congruent polygons with n edges.

12. The pupils in my class work very quickly. Jasleen answers four questions every 30 seconds and Ella answers five questions every 40 seconds.

Last week, Jasleen took exactly 1 hour to answer a large set of questions.

How many minutes more than Jasleen did Ella take to answer the same set of questions?

A 2 B $2\frac{1}{2}$ C $3\frac{1}{4}$ D 4 E $4\frac{1}{2}$

Solution **D**

There are 3600 seconds in an hour. Now $3600 = 120 \times 30$. Since Jasleen answers four questions every 30 seconds in one hour she answers $120 \times 4 = 480$ questions.

Ella answers five questions every 40 seconds. So she takes $40 \div 5$ seconds, that is, 8 seconds per question. Since $480 \times 8 = 3840$, it takes Ella 3840 seconds to answer 480 questions.

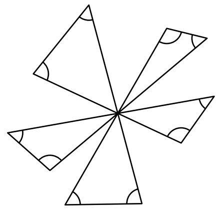
It follows that Ella takes $3840 - 3600 = 240$ more seconds than Jasleen to answer the same set of questions.

Since 240 seconds = 4 minutes, we conclude that Ella takes 4 minutes longer than Jasleen.

13. Five line segments coincide at a point as shown.

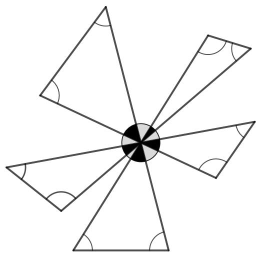
What is the sum of the marked angles?

A 900° B 720° C 540° D 360° E 180°



SOLUTION **B**

We shall refer to the angles marked in the diagram in the question as *white angles*, the other angles in the triangles as *black angles*, and to the angles vertically opposite the black angles as *grey angles*, as shown in the diagram on the right.



The sum of the white angles and the black angles is the sum of all the angles in the five triangles. The sum of the angles in one triangle is 180°. Therefore the sum of the white angles and the black angles is $5 \times 180° = 900°$.

Each black angle is equal to the corresponding vertically opposite grey angle. Therefore the sum of the black angles equals the sum of the grey angles.

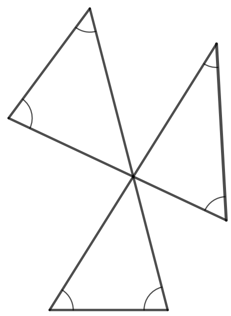
The black angles together with the grey angles are all the angles round the central point. Therefore the sum of the black angles and the grey angles is 360°.

It follows that the sum of the black angles is half of 360°. Therefore this sum is 180°.

We conclude that the sum of the white angles is $900° - 180° = 720°$.

FOR INVESTIGATION

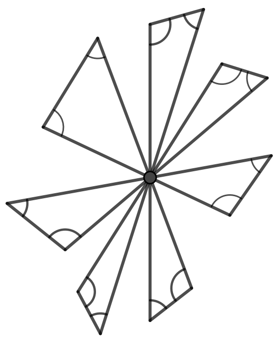
13.1 Three line segments coincide at a point as shown. Find the sum of the marked angles.



13.2

(a) Seven line segments coincide at a point as shown. Find the sum of the marked angles.

(b) Now consider the general case where $2n + 1$ line segments coincide at a point. Find a general formula for the sum of the marked angles in this case.



14. I begin with a three-digit positive integer. I divide it by 9 and then subtract 9 from the answer. My final answer is also a three-digit integer.

How many different positive integers could I have begun with?

A 3 B 5 C 7 D 11 E 13

SOLUTION **A**

When the initial positive integer is divided by 9 and 9 is subtracted, the answer is another integer. Therefore the initial number is a multiple of 9. Therefore we suppose that it is $9n$, where n is some positive integer.

Because the initial number $9n$ is a three digit number, $100 \leq 9n \leq 999$, and therefore

$$12 \leq n \leq 111. \qquad (1)$$

When $9n$ is divided by 9 and 9 is subtracted, the answer is another three-digit number. Therefore $100 \leq n - 9 \leq 999$. Hence

$$109 \leq n \leq 1008. \qquad (2)$$

Combining (1) and (2), the only possible values for n are 109, 110 and 111. Therefore the only possible values for the initial number, $9n$, are 981, 990 and 999. Hence there are 3 possibilities for the initial number.

FOR INVESTIGATION

14.1 I begin with with a four-digit integer. I divide it by 7 and then subtract 7 from the answer. My final answer is also a four-digit integer.

How many different positive integers could I have started with?

15. Alex has a pile of two pence coins. She swapped exactly half of them for the same number of 10p coins. Now she had £4.20.

How much money did Alex have initially?

A 42p B 84p C £1.12 D £1.40 E £1.68

SOLUTION **D**

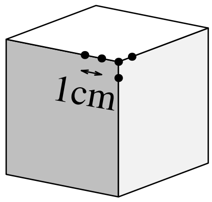
Suppose Alex began with with $2x$ two pence coins. When she swaps half of them for ten pence coins, she has x two pence coins and x ten pence coins. Therefore she has $2x + 10x$ pence, that is $12x$ pence.

She then has £4.20 and therefore $12x = 420$. Hence $x = 35$. It follows that Alex began with 70 two pence coins. Therefore she had £1.40 initially.

FOR INVESTIGATION

15.1 This time, Alex has a different pile of two pence coins. She swapped one fifth them for the same number of 20p coins. Again, she ended up with £4.20. How much money did Alex have in her pile of coins to start with?

16. A cube has edge length 10 cm. Starting at the vertices, dots are placed along every edge at 1 cm intervals. Part of this pattern is shown.



How many dots will there be in total, once the pattern has been completed?

A 128 B 116 C 112 D 108 E 104

SOLUTION **B**

There are 11 dots on each edge. Of these, 2 are at the vertices at the ends of the edge and 9 are in the interior of the edge.

The cube has 12 edges and 8 vertices. Therefore there are $12 \times 9 = 108$ dots in the interiors of edges, and 8 at the vertices.

This makes a total of $108 + 8 = 116$ dots.

17. In 1770, Joseph-Louis Lagrange proved that every positive integer can be written as the sum of four squares. For example, $13 = 0^2 + 0^2 + 2^2 + 3^2$.

How many of the first 15 positive integers can be written as the sum of *three* squares?

A 11 B 12 C 13 D 14 E 15

SOLUTION **C**

COMMENTARY

Instead of trying to write each of the positive integers from 1 to 15 as the sum of three squares, we start with the squares 0, 1, 4 and 9 that are not greater than 15, and work out systematically all the non-zero totals we can make by adding three of them to make a total not greater than 15.

We have

$0^2 + 0^2 + 1^2 = 1,\quad 0^2 + 0^2 + 2^2 = 4,\quad 0^2 + 0^2 + 3^2 = 9,\quad 0^2 + 1^2 + 1^2 = 2,$

$0^2 + 1^2 + 2^2 = 5,\quad 0^2 + 1^2 + 3^2 = 10,\quad 0^2 + 2^2 + 2^2 = 8,\quad 0^2 + 2^2 + 3^2 = 13,$

$1^2 + 1^2 + 1^2 = 3,\quad 1^2 + 1^2 + 2^2 = 6,\quad 1^2 + 1^2 + 3^2 = 11,\quad 1^2 + 2^2 + 2^2 = 9,$

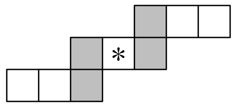
$1^2 + 2^2 + 3^2 = 14,\quad 2^2 + 2^2 + 2^2 = 12.$

We see that the positive integers in the range from 1 to 15 that can be written as the sum of three squares are

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14.$$

Therefore there are 13 integers in the range from 1 to 15 that can be written as the sum of three squares.

18. Each of the numbers 1 to 9 is to be placed in a different cell of the grid shown so that the sum of the three numbers in each row is 15. Also, the sum of the two numbers in each shaded column is to be 15.

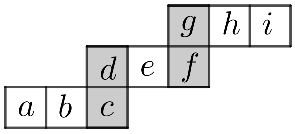


How many choices are there for the number to be placed in the central cell indicated by *?

A 0 B 1 C 2 D 3 E 4

SOLUTION **C**

For convenience, we have labelled the numbers in the cells as shown in the diagram on the right.



We first consider the possible values for c, d, f and g.

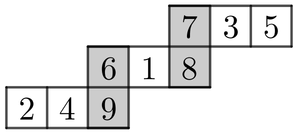
We need to have $c + d = 15 = f + g$. The only pairs of integers in the range from 1 to 9 with sum 15 are $7 + 8$ and $6 + 9$.

We cannot have $d = 9$, because in that case f is either 7 or 8. In either case, $d + f$ would be greater than 15, and so $d + e + f \neq 15$. Similarly, we cannot have $f = 9$. Therefore either d or f must be 6.

If $d = 6$ then f is either 7 or 8. Both $d = 6$, $e = 1$, $f = 8$, and $d = 6$, $e = 2$, $f = 7$ meet the requirement that $d + e + f = 15$. Thus if $d = 6$, the only possible values for e are 1 and 2. We can draw the same conclusion if $f = 6$.

It remains only to show that both these possible values for e are compatible with the condition that the sum of the numbers in each row is 15.

The diagram on the right shows that it is possible to meet this condition with $e = 1$.



It is left for the reader to check that it is also possible to meet this condition with $e = 2$.

Therefore there are 2 possible choices for the number to be placed in the central cell marked *.

FOR INVESTIGATION

18.1 Check that it is possible to meet the condition of the question with $e = 2$.

18.2 In how many different ways is it possible to place the integers from 1 to 9, with one number in each cell, so that the sum of the numbers in each row and in each of the shaded columns is 15?

18.3 Is it possible to place the integers from 1 to 9, with one number in each cell, so that the sums of the numbers in each row and each column are all the same, but with a common total other than 15?

19. In my class, everyone studies French or German, but not both languages. One third of the girls and the same number of boys study German. Twice as many boys as girls study French.
Which of these could be the total number of boys and girls in my class?

A 26 B 28 C 30 D 32 E 34

SOLUTION **D**

To avoid fractions, we let $3k$ be the number of girls.

One third of the girls study German. Therefore k girls study German and $2k$ girls study French.

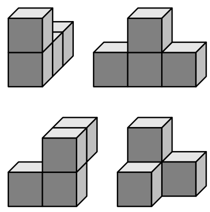
The number of boys studying German is the same as the number of girls studying German. Therefore k boys study German.

The number of boys studying French is twice the number of girls studying French. Therefore $4k$ boys study French.

It follows that the total number of boys and girls in the class is $k + 2k + k + 4k = 8k$. Since k is an integer, the number of girls and boys in the class is a multiple of 8.

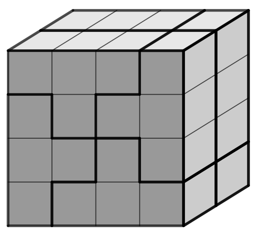
Of the given options only 32 is a multiple of 8. Therefore it is the only option which could be the total number of boys and girls in the class. This case is possible with 12 girls in the class of whom 8 study French, and 20 boys, of whom 16 study French.

20. Each of the shapes shown has been made from four unit cubes. For each shape, Max takes eight copies of the shape and tries to fit them together to make a $2 \times 4 \times 4$ cuboid.
How many of the shapes can be used to make a cuboid of this size in this way?



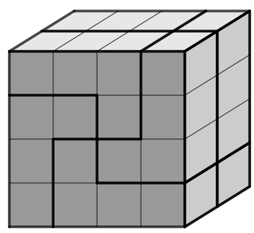
A 0 B 1 C 2 D 3 E 4

SOLUTION **E**



The diagrams on the right show how eight copies of each of the shapes in the top row may be assembled to make a $2 \times 4 \times 4$ cuboid.

It can be seen that in each case it is possible to use four of the given shapes to make a $1 \times 4 \times 4$ cuboid. Thus eight of the shapes may be used to make two of these cuboids. These two cuboids may then be put together to make a $2 \times 4 \times 4$ cuboid.



It may be seen that two copies of each of the shapes in the bottom row may be put together to make a 2×2 cuboid. It is not easy to illustrate this with a diagram, so we leave it to the reader to either visualize this or check it by making the relevant shapes. Then four of these $2 \times 2 \times 2$ cuboids may be put together to make a $2 \times 4 \times 4$ cuboid.

It follows each of the four given shapes may be used to construct a $2 \times 4 \times 4$ cuboid.

21. Some fish, some dogs and some children are swimming in a bay. There are 40 legs in total, twice as many heads as tails and more dogs than fish.

How many fish are in the bay?

A 1 B 2 C 3 D 4 E 5

Solution **B**

We let f, d and c be the numbers of fish, dogs and children, respectively. From the wording of the question we deduce that f, d and c are all positive integers.

Each dog has four legs, each child has two legs and the fish do not have legs. Therefore, from the fact that there are 40 legs in total, we have

$$4d + 2c = 40. \qquad (1)$$

Each fish, dog and child has one head. Therefore there are $f + d + c$ heads. Each fish and each dog has one tail. Therefore there are $f + d$ tails. Hence, from the fact that there are twice as many heads as tails, we have

$$f + d + c = 2(f + d). \qquad (2)$$

By (2)

$$c = f + d. \qquad (3)$$

Substituting from (3) in (1) gives

$$4d + 2(f + d) = 40. \qquad (4)$$

Equation (4) may be rearranged as

$$6d + 2f = 40. \qquad (5)$$

It follows that

$$3d + f = 20. \qquad (6)$$

Since there are more dogs than fish, $3d + f < 3d + d = 4d$. Hence, by (6),

$$20 < 4d \qquad (7)$$

and hence

$$5 < d. \qquad (8)$$

Because $f > 0$, it follows from (6), that $3d < 20$ and hence

$$d < 7. \qquad (9)$$

It follows from (8) and (9) that $d = 6$.

Therefore, by (6), $f = 20 - 3 \times 6 = 2$.

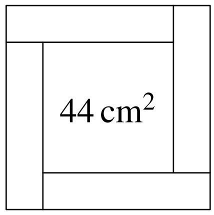
For investigation

21.1 How many children are there in the bay?

21.2 Suppose that there are 40 legs in total, three times as many heads as tails and again more dogs than fish.

How many children, dogs and fish are there in the bay?

22. The diagram shows four congruent rectangles, each of perimeter 20 cm, surrounding a square of area 44 cm^2.

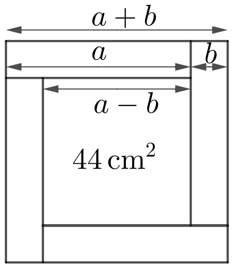


What is the area of each rectangle?

A 14 cm^2 B 16 cm^2 C 18 cm^2 D 20 cm^2 E 22 cm^2

SOLUTION **A**

We let the length of each rectangle be a cm and the width of each rectangle be b cm.



Each rectangle has two sides of length a cm, and two sides of length b cm. Therefore the perimeter of each rectangle is $(2a + 2b)$ cm. It follows that $2a + 2b = 20$. Therefore

$$a + b = 10.$$

We see from the diagram that the outer square has side length $(a + b)$ cm. Hence the outer square has area 10^2 cm^2, that is, 100 cm^2.

The total area of the four rectangles is the difference between the areas of the outer and inner squares.

Hence this area is 100 cm^2 − 44 cm^2 = 56 cm^2.

Therefore the area of each of the four rectangles is $(56 \div 4)$ cm^2 = 14 cm^2.

FOR INVESTIGATION

22.1 Suppose that the area of each rectangle is 25 cm^2 and, as in the question, the area of the inner square is 44 cm^2.

What is the perimeter of each rectangle?

22.2 We have seen from the solution that

$$(a + b)^2 = 100. \qquad (1)$$

We see from the diagram in the solution that

$$(a - b)^2 = 44. \qquad (2)$$

Use equations (1) and (2) to find the value of the product ab.

22.3 Suppose that

$$(a + b)^2 = 143,$$

and

$$(a - b)^2 = 75.$$

What is the value of the product ab?

23. Four different positive integers p, q, r, s satisfy the equation

$$(9-p)(9-q)(9-r)(9-s) = 9.$$

What is the value of $p+q+r+s$?

A 20 B 24 C 28 D 32 E 36

SOLUTION **E**

COMMENTARY

This is a perplexing question until you have an "aha" moment.

Because p, q, r and s are different positive integers, the factors $(9-p)$, $(9-q)$, $(9-r)$ and $(9-s)$ are all different. However, the only way to express 9 as the product of different positive integers is as 1×9. So 9 is not the product of four different positive integers.

But, although p, q, r and s are positive, not all the factors $(9-p)$, $(9-q)$, $(9-r)$ and $(9-s)$ need be positive. Aha!

There is just one set of four different integers whose product is 9, namely -3,-1, 1 and 3.

Therefore the values of $(9-p)$, $(9-q)$, $(9-r)$ and $(9-s)$ are -3, -1, 1 and 3 in some order.

Hence p, q, r and s are 12, 10, 8 and 6 in some order.

Therefore the value of $p+q+r+s$ is $12+10+8+6=36$.

FOR INVESTIGATION

23.1 Four different positive integers p, q, r, s satisfy the equation

$$(9-p)(9-q)(9-r)(9-s) = 49.$$

What is the value of $p+q+r+s$?

23.2 Five different positive integers p, q, r, s, t satisfy the equation

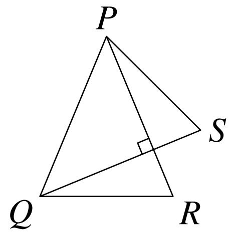
$$(9-p)(9-q)(9-r)(9-s)(9-t) = 12.$$

What is the value of $p+q+r+s+t$?

24. In the diagram shown, $PQ = PR = QS$. Line segments PR and QS are perpendicular to each other.

What is the sum of $\angle PRQ$ and $\angle PSQ$?

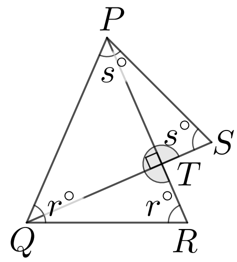
A 90° B 105° C 120° D 135° E 150°



SOLUTION **D**

Let T be the point where PR meets QS.

Let $\angle PRQ = r°$ and $\angle PSQ = s°$.



Because $PQ = PR$, the triangle PQR is isosceles, and therefore

$$\angle PQR = \angle PRQ = r°.$$

Because $PQ = QS$, the triangle QPS is isosceles, and therefore

$$\angle SPQ = \angle PSQ = s°.$$

Because PR is perpendicular to QS, the marked reflex angle RTS is 270°.

The sum of the interior angles of a pentagon is 540°. Therefore from the pentagon $PQRTS$ we have

$$r + r + s + s + 270 = 540.$$

Therefore

$$2r + 2s = 540 - 270 = 270,$$

and hence

$$r + s = 135.$$

Therefore the sum of $\angle PRQ$ and $\angle PSQ$ is 135°.

FOR INVESTIGATION

24.1 (a) Find $\angle TPS$ in terms of r and s.

(b) Use the fact that the sum of the angles in the triangle TPS is 180° to find the value of $r + s$.

24.2 Suppose that $\angle RPS = 15°$. Find $\angle PRQ$ and $\angle PSQ$ in this case.

24.3 The solution above uses the fact that the sum of the interior angles of a pentagon is 540°. Explain why this is correct.

24.4 (a) Find a formula, in terms of n, for the sum of the interior angles of a polygon with n vertices.

(b) Check that your formula gives the value 540° when $n = 5$.

25. I choose four different integers. When I add all the pairs of these numbers in turn, the totals that I obtain are 23, 26, 29, 32 and 35, with one of these totals being repeated. What is the largest of the four integers?

A 18 B 19 C 20 D 21 E 22

Solution **B**

Let the four different integers be a, b, c and d. Suppose that $a < b < c < d$.

It follows that

$$a + b < a + c < b + c < b + d < c + d$$

and

$$a + b < a + c < a + d < b + d < c + d.$$

We deduce that the sums $a + b$, $a + c$, $b + d$, and $c + d$ are all different, but that $a + d = b + c$.

The totals of pairs of these numbers satisfy

$$23 < 26 < 29 < 32 < 35.$$

It follows that

$$a + b = 23, \qquad (1)$$
$$a + c = 26, \qquad (2)$$
$$a + d = b + c = 29, \qquad (3)$$
$$b + d = 32, \qquad (4)$$

and

$$c + d = 35. \qquad (5)$$

By (4) and (5)

$$\begin{aligned} b + c + 2d &= (b + d) + (c + d) \\ &= 32 + 35 \\ &= 67. \qquad (6) \end{aligned}$$

From (3) and (6)

$$\begin{aligned} 2d &= (b + c + 2d) - (b + c) \\ &= 67 - 29 \\ &= 38. \end{aligned}$$

It follows that $d = 19$. Therefore the largest of the four integers is 19.

For investigation

25.1 Find the values of a, b and c.

25.2 I choose five different integers. When I add pairs of these integers in turn, the totals that I obtain are 29, 31, 35, 36, 40, 42, 45, 47 and 51, with one of the totals being repeated.

Which are my five integers?